Singular points and Zeros

• Isolated singular pt.

z_0 is called isolated singular pt. if there some open neighbourhood of it throughout $f(z)$ is analytic except at z_0 only.

• Classification of isolated singular points

• Poles

if we can find apositive integer "n" ∍

$$\lim_{z\to z_0} (z-z_0)^n f(z) \neq 0$$

then z_0 is called apole of order "n". If $n=1$ z_0 is asimple pole

• لوفي Lau.Ser. كانهناك حدود
+ve power و شويه -ve power

• Removable singular pt.

"z_0" is called a removable s.p of $f(z)$ if

$$\lim_{z\to z_0} f(z) \text{ exist}$$

• لو Lau.S كانت كلها
حدود +ve power
فقط

• essential s.p

"z_0" is e.s.p. if it isnot a pole or removable

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analytic
Part

• Zeros

A point z_0 is called a zero of order n for the function f if f is analytic at z_0 and

$$f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(n-1)}(z_0) = 0 \neq f^{(n)}(z_0)$$

• ex $f(z) = (z-5)^3$

$f(5) = f'(5) = f''(5) = 0$ but $f'''(5) = 6$

thus $z = 5$ is zero of order 3.

1- Determine the zeros for the function

$$F(z) = z \sin z^2$$

Sol.

$\because \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \Rightarrow \sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots$

$$\therefore F(z) = z \sin z^2 = z\left(z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots\right)$$

$$\therefore F(z) = z^3\left(1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \dots\right)$$

• Hence

$z=0$ is zero of order 3.

• Pole of order "n"

If functions f and g are analytic at $z=z_0$ and f has a zero of order n at $z=z_0$ and $g(z_0) \neq 0$ then the function

$$F(z) = \frac{g(z)}{f(z)}$$

has a pole of order "n" at "$z=z_0$"

2- classify the singularity at $z=1$ of the function

$$F(z) = \frac{\sin z}{(z^2-1)^2}$$

Sol.

$$F(z) = \frac{\sin z}{((z-1)(z+1))^2} = \frac{\sin z/(z+1)^2}{(z-1)^2}$$

The numerator is analytic and nonzero at $z=1$, then the function has a pole of order 2 at $z=1$.

3- Locate the isolated singular points of the following functions.

- $f(z) = \dfrac{z - \sin z}{z^3}$

Sol

$z=0$ is isolated singular point

$$\therefore \lim_{z \to 0} \frac{z-\sin z}{z^3} = \lim_{z\to 0} \frac{1-\cos z}{3z^2} = \lim_{z \to 0} \frac{\sin z}{6z}$$

$$= \lim_{z\to 0} \frac{\cos z}{6} = \frac{1}{6}$$

$\therefore$ $z=0$ is removable.

- $f(z) = e^{1/z}$

Sol.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$\therefore$ z_0 is essential.

- $f(z) = \sec(\frac{1}{z})$

Sol.

$$f(z) = \frac{1}{\cos(\frac{1}{z})}$$

$$\cos(\tfrac{1}{z}) = 0 \Rightarrow \frac{1}{z} = (2n+1)\frac{\pi}{2}$$

$\therefore$ $z = \dfrac{2}{(2n+1)\pi}$ is a simple poles.

$n = 0, \pm 1, \pm 2, \ldots$

also, since $f(z)$ isn't defined at $z=0 \Rightarrow z=0$ is a singular pt.

The Residue Theorem and its Applications

• Calculation of Residues

• Residue at a simple pole

if $f(z)$ has a simple pole at $z = z_0$, then

$$Res(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

• Residue at a pole of order "n"

if $f(z)$ has a pole of order "n" at $z = z_0$, then

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \cdot \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

1- Find the residues at singularities of

a- $f(z) = \dfrac{e^z}{z(z+1)}$

Sol.

• Singular pts. $z = 0$, $z = -1$

at $z=0$

$$Res(f(z), 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^z}{(z+1)} = 1$$

$$Res(f(z), 0) = 1$$

at $z=1$

$$Res(f(z), -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{e^z}{z} = \frac{-1}{e}$$

$$\therefore Res(f(z), -1) = \frac{-1}{e}$$

b- $F(z) = \dfrac{\cos z}{z^2(z-\pi)^3}$

Sol.

• $F(z)$ has a pole of order "2" at $z=0$ and a pole of order "3" at $z=\pi$.

at $z=0$

$$\text{Res}(F(z),0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\, z^2 F(z) = \lim_{z\to 0}\frac{d}{dz}\frac{\cos z}{(z-\pi)^3}$$

$$= \lim_{z\to 0}\frac{-(z-\pi)\sin z - 3\cos z}{(z-\pi)^4} = -\frac{3}{\pi^4}$$

$$\therefore \text{Res}(F(z),0) = \frac{-3}{\pi^4}$$

at $z=\pi$

$$\text{Res}(F(z),0) = \frac{1}{2!}\lim_{z\to\pi}\frac{d^2}{dz^2}(z-\pi)^3 F(z)$$

$$= \frac{1}{2}\lim_{z\to\pi}\frac{d^2}{dz^2}\frac{\cos z}{z^2}$$

$$= \frac{1}{2}\lim_{z\to\pi}\frac{(6-z^2)\cos z + 4z\sin z}{z^4} = -\frac{6-\pi^2}{2\pi^4}$$

$$\text{Res}(F(z),\pi) = -\frac{6-\pi^2}{2\pi^4}$$

2- Evaluate the residue of

$$F(z) = \frac{1+z}{1-\cos z} \quad \text{at the origin}$$

Sol.

$$F(z) = \frac{1+z}{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)}$$

$$\therefore F(z) = \frac{2(1+z)}{z^2\left(1-\frac{z^2}{12}+\cdots\right)}$$

Thus $F(z)$ has a pole of order "2" at $z=0$

$$\therefore \operatorname{Res}(f(z),0) = \lim_{z\to 0} \frac{d}{dz} z^2 f(z)$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{2(1+z)}{(1-\frac{z^2}{12}+\cdots)}\right]$$

$$= \lim_{z\to 0}\left[\frac{2(1-\frac{z^2}{12}+\cdots)-(1+z)(-\frac{z}{6}+\cdots)}{(1-\frac{z^2}{12}+\cdots)^2}\right] = 2$$

$$\therefore \operatorname{Res}(f(z),0) = 2$$

• An alternative method for computing a residue at a Simple Pole.

Suppose a function $f(z)$ can be written as a quotient

$$f(z) = \frac{g(z)}{h(z)}$$

where "g" and "h" are analytic at "$z = z_0$". If $g(z_0) \neq 0$ and if the function "h" has a zero of order "1" at z_0, then $f(z)$ has a Simple pole at "$z = z_0$" and

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

3- Compute the residue at each Singularity of $f(z) = \frac{3z^2+1}{z^2+1}$

Sol.

• $f(z)$ has two Simple poles at $z = i$ and $z = -i$

$z = i$

$$\operatorname{Res}(f(z), i) = \frac{3(i)^2+1}{2(i)} = \frac{-2}{2i} = \frac{-1}{i} = i$$

$$\therefore \operatorname{Res}(f(z), i) = i$$

$z = -i$

$$\operatorname{Res}(f(z), -i) = \frac{3(-i)^2+1}{2(-i)} = \frac{-2}{-2i} = \frac{1}{i} = -i$$

$$\operatorname{Res}(f(z), -i) = -i$$

Note

The Coefficient a_{-1} in Laurent Series is Called the residue of $f(z)$ at z_0 and is denoted by $Res(f(z), z_0)$.

· Cauchy's Residue theorem

Let D be a Simple Connected domain and C a Simple closed Contour lying entirely within D. If a function f is analytic on and inside C, except at a finite number of Singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

4- Evaluate

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$$

where

1- the Contour C is the rectangle defined by $x=0$, $=4$ $y=0$ and $y=-1$

2- the Contour C is the Circle $|z|=2$

Sol.

1- Since both poles $z=1$ and $z=3$ lie within the rectangle then

$$I = 2\pi i\,(Res(f(z),1) + Res(f(z),3))$$

$z=1$

$$Res(f(z),1) = \lim_{z\to 1} \frac{d}{dz}\frac{1}{(z-3)} = \frac{-1}{4}$$

$z=3$

$$Res(f(z),3) = \lim_{z\to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$$

$$\therefore I = 0$$

2- only the pole $z=1$ lie within the Circle $|z|=2$

thus

$$I = 2\pi i \,(\text{Res}(f(z),1))$$

$$\therefore \boxed{I = \frac{-\pi i}{2}}$$

5- Evaluate

$$\oint_C \frac{e^z}{z^4+5z^3}\,dz$$

$C: |z| = 2$

Sol.

$f(z)$ has asimple pole at $z=5$ and apole of order "3" at $z=0$

Since only $z=0$ lies within the Contour, then

$$I = 2\pi i \,\text{Res}(f(z),0)$$

$$= 2\pi i \left[\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{e^z}{z+5}\right)\right]$$

$$= \pi i \lim_{z\to 0}\frac{(z^2+8z+17)\,e^z}{(z+5)^3}$$

$$\boxed{I = \frac{17\pi}{125}\,i}$$

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